Perfectly Matched Layers equations for 3D acoustic wave propagation in heterogeneous media

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Abstract

This work is dedicated to the analysis of Berenger PML method applied to the 3D linearized Euler equations without advection terms, with variable wave velocity and acoustic impedance. It is an extension of a previous work presented in a 2D context [8]. The 3D linearized Euler equations are used to simulate propagation of acoustic waves beneath the subsurface. We propose an analysis of these equations in a general heterogeneous context, based on *a priori* error estimates. Following the method introduced by Métral and Vacus [9], we derive an augmented system from the original one, involving the primitive unknowns and their first order spatial derivatives. We define a symetrizer for this augmented system. This allows to compute energy estimates in the three following cases: the Cauchy problem, the half-space problem with a non homogeneous Dirichlet boundary condition and finally the transmission problem between two half-spaces separated by an impedance discontinuity.

Introduction

Geophysicists are interested in the modeling of seismic wave propagation beneath the subsurface. To this purpose, for instance in seismic imaging, they can be led to use acoustic wave propagation model. In this context, wave propagation is described by the following equation

$$\partial_{tt}p(\mathbf{x},t) - c(\mathbf{x})I(\mathbf{x})\text{div}\left(\frac{c(\mathbf{x})}{I(\mathbf{x})}\nabla p(\mathbf{x},t)\right) = 0, \qquad (1)$$

where

- $\mathbf{x} = (x,y,z) \in R^3$ is the vector of spatial coordinates;
- $t \in [0,T]$ is the time variable;
- $p(\mathbf{x},t)$ is the pressure wavefield;
- $c(\mathbf{x})$ is the wave velocity;
- $I(\mathbf{x})$ is the acoustic impedance;
- div is the divergence operator;
- ∇ is the gradient operator.

Using absorbing boundary conditions is essential if one wants to avoid fictitious reflections at the computational domain boundaries. To this purpose, the PML (Perfectly Matched Layers) method has been designed by Bérenger [2], initially for the 2D Maxwell equations. The principle of absorbing layers is to surround the computational domain with a non zero width layer in which the incident waves should decay. This layer must also produce as weak as possible reflections at the interface with the interest domain. The PML method fulfill these two requirements. Since its introduction in 1994, it has been successfully applied to numerous physical domains: electromagnetic, acoustic and elastic wave propagation for instance. The popularity of this method mostly comes from its efficiency and its ease of implementation. On a mathematical point of view, it consists in a two steps modification of the initial system of equations:

- first, split the unknowns in the space directions;
- second, introduce absorption terms which are zero in the interest domain and positive in the layer.

The most important drawback of the Bérenger PML method is the potential loose of the well-posedness that may occur by changing the form of the initial system of equations. An intensive analysis of this method is proposed by Halpern, Petit-Bergez and Rauch in [3]. The authors give a theoretical framework allowing to extend the PML method to all kind of general first order systems of hyperbolic partial differential equations, and provide tools to analyse the well-posedness of the resulting systems. In particular, it is proved that for an initial well-posed system, the PML corresponding system is at least weakly-well posed, and strongly well posed if the principal part of the symbol of the partial operator is elliptic. They also propose a new method they call HML (Harmoniously Matched Layers) that keep the well-posedness of the equations and cancel reflexion coefficients to the first order.

Here, we focus on the analysis of the Bérenger PML equations by means of energy method. Indeed, in order to apply the PML method to the equation (1), we rewrite it as the following first order hyperbolic system

$$\left\{\begin{array}{l} \partial_t\mathbf{u}(\mathbf{x},t) - \frac{c(\mathbf{x})}{I(\mathbf{x})}\nabla p(\mathbf{x},t) = 0 \\ \partial_t p(\mathbf{x},t) - c(\mathbf{x})I(\mathbf{x})\ \text{div}\ \mathbf{u}(\mathbf{x},t) = 0. \end{array}\right. \qquad (2)$$

Here $\mathbf{u}(\mathbf{x},t)$ is the velocity displacement field. These equations are equivalent to the linearized Euler equations, without advection. The PML equations for the 2D linearized Euler equations have been studied by Hesthaven [4], and Hu [5], with advection, in a context of homogeneous media: the wave velocity and the acoustic impedance are constants ($c(\mathbf{x}) = c_0$ and $I(\mathbf{x}) = I_0$). Hesthaven shows that, in this context, the PML system is weakly well-posed, which is an explanation of numerical instabilities evidenced by Hu. Hesthaven also proposes a modified PML system which preserves the well-posedeness of the initial Euler equations. Unfortunately, this method shows up to be less efficient numerically.

Here, we are interested in modeling acoustic wave propagation in 3D heterogeneous media, for which both velocity field and acoustic impedance depend on spatial coordinates $\mathbf{x}$. Furthermore, instead of splitting all the unknowns, we rather prefer to split only the pressure field $p(\mathbf{x},t)$ in three components $p_x(\mathbf{x},t), p_y(\mathbf{x},t), p_z(\mathbf{x},t)$ such that

$$p(\mathbf{x},t) = p_x(\mathbf{x},t) + p_y(\mathbf{x},t) + p_z(\mathbf{x},t) \tag{3}$$

The solution of the partially split PML system is equivalent to the solution of the totally split system [3]. This allows us to reduce the number of unknowns from 12 to 6, thus reducing the computation cost. We obtain the following PML system

$$\left\{\begin{array}{l}
\partial_t u_x(\mathbf{x},t) - \frac{c(\mathbf{x})}{I(\mathbf{x})}\partial_x p(\mathbf{x},t) + \sigma_x(x) u_x(\mathbf{x},t) = 0\\
\partial_t u_y(\mathbf{x},t) - \frac{c(\mathbf{x})}{I(\mathbf{x})}\partial_y p(\mathbf{x},t) + \sigma_y(y) u_y(\mathbf{x},t) = 0\\
\partial_t u_z(\mathbf{x},t) - \frac{c(\mathbf{x})}{I(\mathbf{x})}\partial_z p(\mathbf{x},t) + \sigma_z(z) u_z(\mathbf{x},t) = 0\\
\partial_t p_x(\mathbf{x},t) - c(\mathbf{x})I(\mathbf{x})\partial_x u_x(\mathbf{x},t) + \sigma_x(x) p_x(\mathbf{x},t) = 0\\
\partial_t p_y(\mathbf{x},t) - c(\mathbf{x})I(\mathbf{x})\partial_y u_y(\mathbf{x},t) + \sigma_y(y) p_y(\mathbf{x},t) = 0\\
\partial_t p_z(\mathbf{x},t) - c(\mathbf{x})I(\mathbf{x})\partial_z u_z(\mathbf{x},t) + \sigma_z(z) p_z(\mathbf{x},t) = 0
\end{array}\right. \tag{4}$$

where $\sigma_x(x), \sigma_y(y), \sigma_z(z)$ are absorption coefficients, respectively in the directions x, y and z.

These coefficients are chosen in accordance with the ones chosen by Hu [5]. Basically, they are zero in the interest domain, and grow polynomially from the interface between the computational domain and the layer toward the external border of the layer.

We propose a numerical analysis of this system based on the energy estimates method. Following the work proposed by Métral and Vacus [9], and generalized by Halpern, Petit-Bergez and Rauch [3], we show that for the Cauchy problem, a combination of the L^2 norm and the H^1 norm of the unknowns can be controlled by a combination of the L^2 norm and the H^1 norm of the initial condition. We then extend this result to the two following cases:

- first, we consider a problem defined on a half-space $R^2 \times [0, +\infty[$ with a non homogeneous Dirichlet boundary condition $h(x,y,t)$ on plane $z = 0$, with regular wave velocity $c(\mathbf{x}) \in C^\infty\left(R^2 \times [0,+\infty[\right)$ and acoustic impedance $I(\mathbf{x}) \in C^\infty\left(R^2 \times [0,+\infty[\right)$;
- second, we consider a transmission problem between two half-spaces separated by an impedance discontinuity on the plane $z = 0$, with regular wave velocity $c(\mathbf{x}) \in C^\infty(R^3)$

The Cauchy problem is general and is to be considered as a starting point of this work. The second case arises in application to geophysics when modeling acoustic wave response in a well, knowing the pressure condition at the top of the well. The third case is also interesting for geophysicists because of the intrinsic discontinuity of acoustic impedance.

Preliminary : definition of an augmented system and its symetrizer

We consider an augmented system derived from (4) by adding equations on each of the spatial first derivative of the primitive unknowns $\mathbf{u}$ and p. We obtain a system of 18 equations that may be written as

$$\begin{array}{l}\partial_t U(\mathbf{x},t) - A(\mathbf{x})\partial_x U(\mathbf{x},t) - B(\mathbf{x})\partial_y U(\mathbf{x},t)\\ -C(\mathbf{x})\partial_z U(\mathbf{x},t) + S(\mathbf{x})U(\mathbf{x},t) = 0\end{array} \tag{5}$$

We recall here the Kreiss symetrizer definition [6]: *For a PDE operator*

$$\mathcal{P}(\mathbf{x},t,\partial_x) = \sum_{j=1}^{J} A_j(\mathbf{x},t)\partial_{x_j} \textit{ with symbol}$$

$$P(x,t,i\omega) = i\sum_{j=1}^{J}\omega_j A_j(x,t),$$

a symetrizer of this operator is a hermitian matrix $H(\mathbf{x},t,\omega)$ positive definite for all $\mathbf{x}$, $t \in [0,T]$, ω, $||\omega|| = 1$, C^∞ with respect to all its arguments, which verifies

$$H(\mathbf{x},t,\omega)P(\mathbf{x},t,\omega) + P^*(\mathbf{x},t,\omega)H(\mathbf{x},t,\omega) = 0.$$

We thus propose a first lemma :

Lemma 1 : *There exists a symetrizer $H(\mathbf{x})$ for the operator $\mathcal{P}_1 = A\partial_x + B\partial_y + C\partial_z$. Since $H(\mathbf{x})$ does not depend on ω, it symetrizes simultaneously the matrixes A, B, and C.*

This result is the keystone of the following theorems. Its proof is obtained by exhibiting such a symetrizer, which can thus be used to obtain energy estimates in the three cases of interest.

Case 1 : Cauchy problem for regular coefficients

This case is the easiest one. Using the scalar product induced by $H(\mathbf{x})$, denoted by $(.,.)_H$, we study the quantity $\frac{d}{dt}(U,U)_H$ where U is solution of (5). We note $\Omega = R^3$. Using integration by part and Gronwall lemma, we obtain an estimate that may be written as

$$\begin{array}{l}\|(u_x,u_y,u_z,p\|^2_{L^\infty(0,T;H^1(\Omega))} + \|p_z\|^2_{L^\infty(0,T;L^2(\Omega))} + \\ \|(c(\sigma_x p_x + \sigma_y p_y))\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \\ C_1 e^{C_2 T}\big(\|(u_x,u_y,u_z,p)(.,.,.,0)\|^2_{H^1(\Omega)} + \\ \|c(\sigma_x p_x + \sigma_y p_y))(.,.,.,0)\|^2_{L^2(\Omega)} + \\ \|p_z(.,.,.,0)\|^2_{L^2(\Omega)}\big)\end{array} \tag{6}$$

where C_1 and C_2 depend on $I(\mathbf{x})$, $c(\mathbf{x})$, $\sigma_x(x), \sigma_y(y)$ and $\sigma_z(z)$.

This estimate shows that the H^1 norm of the solution $(\mathbf{u}, p_x, p_y, p_z)$ of the primitive system (4) is controlled by the H^1 norm of the initial state plus additive L^2 norm terms of the initial state. Hence, if the initial state is regular enough, the solution remains bounded in the time interval $[0, T]$ with a potential exponential growth of its H^1 norm. Moreover, this ensures the unicity of the solution. Following the work of Petit-Bergez [10], the question of the existence may be adressed deriving the same kind of estimate on the semi-discretized in space problem. This estimate should be uniform as the discretization parameter tends to zero. Passing to the limit should prove the existence.

Case 2 : half-space problem with non homogeneous Dirichlet boundary condition and regular wave velocity and acoustic impedance

Proceeding in this case in the same way as in the previous one, a difficulty arises: the integration by part of the quantity $\frac{d}{dt}(U,U)_H$ following the z direction induces a non-zero boundary term at the border $z = 0$. This boundary term prevents from obtaining directly the energy estimates, as in the previous case.

However, making repeated use of the lifting theorem introduced by Lions and Magenes [7], it is possible to come back to the previous situation. We denote $\Omega = R^2 \times [0, +\infty[$ and $\Sigma = R^2$. We thus obtain the following energy estimates, depending on the regularity of the boundary condition in spaces $H^{r,s}(\Sigma)$:

$$\begin{array}{l}\|(u_x,u_y,u_z,p\|^2_{L^\infty(0,T;H^1(\Omega))} + \|p_z\|^2_{L^\infty(0,T;L^2(\Omega))} + \\ \|(c(\sigma_x p_x + \sigma_y p_y))\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \\ C_1\big(\|(u_x,u_y,u_z,p)(.,.,.,0)\|^2_{H^1(\Omega)} + \\ \|c(\sigma_x p_x + \sigma_y p_y))(.,.,.,0)\|^2_{L^2(\Omega)} + \\ \|p_z(.,.,.,0)\|^2_{L^2(\Omega)} + \big) \\ C_2\|h\|^2_{H^{7/2,7/2}(\Sigma)} + C_3\|h\|^2_{H^{7/2,7/2}(\Sigma)}\end{array} \tag{7}$$

where C_1, C_2 and C_3 depend on $I(\mathbf{x})$, $c(\mathbf{x})$ and $\sigma_x(x), \sigma_y(y), \sigma_z(z)$.

In this case, the H^1 norm of the solution $(\mathbf{u}, p_x, p_y, p_z)$ of the primitive system (4) is controlled by the sum of the H^1 norm of the initial state and additive L^2 norm terms of the initial state plus the norm of the boundary condition on the trace space $H^{7/2,7/2}(\Sigma)$.

Hence, if the boundary condition $h(x, y, t)$ is regular enough, the norm of the solution of the primitive system remains bounded on the intervalle $[0, T]$ with a potential exponential growth. This estimate ensures the unicity of the solution, and the existence problem may be adressed again studying the semi-discretized in space problem.

Case 3 : Transmission problem with acoustic impedance discontinuity on plane $z = 0$

In this case we consider two half-spaces

$$\Omega_1 = R^2 \times]-\infty, 0],\ \Omega_2 = R^2 \times [0, +\infty[.$$

We define the corresponding acoustic impedance functions $I_1(\mathbf{x})$ on Ω_1, $I_2(\mathbf{x})$ on Ω_2, such that

$$\forall (x,y) \in R^2,\ I_1(x,y,0^-) \neq I_2(x,y,0^+)$$

We suppose that at time $t = 0$ there exists a solution $(u_x(\mathbf{x},0), u_z(\mathbf{x},0), p_x(\mathbf{x},0), p_z(\mathbf{x},0))$ of the equations (4) on the whole space $\Omega = \Omega_1 \cup \Omega_2$. We are interested in finding energy estimates for this solution on the time intervalle $[0, T]$.

We proceed in the same way as for the other cases: on each half space Ω_i we define a symetrizer H_i of the augmented system (5) and we study the quantities $\frac{d}{dt}(U_i, U_i)_{H_i}$. Integrations by part following z direction yield two boundary terms that prevent from direct application of Gronwall lemma. The difference between these two terms is what is called the ”jump” term in the study of transmission problem.

In order to obtain the desired energy estimate on the solution on Ω, it is necessary to prove that this jump term is bounded. In a context of $3D$ acoustic impedance function

$I_i(\mathbf{x})$, this seems to remain a difficult problem. However, assuming $1D$ acoustic impedance functions $I_i(z)$ such that

$$I_1(0^-) \neq I_2(0^+)$$

and using the continuity of the following quantities at the interface $z = 0$:

- wave velocity $c(\mathbf{x})$;
- velocity displacement normal to the interface $u_z(\mathbf{x}, t)$;
- pressure wavefield $p(\mathbf{x}, t)$;

we prove that the jump term is zero. Thus, in this more restrictive context of 1D acoustic impedance, we obtain again the energy estimate (6).

Hence, for a regular enough initial condition defined on Ω, assuming that the impedance function depends only on the vertical direction z, the H^1 norm of the solution of the primitive system (4) is controlled by the H^1 norm of the initial state plus additive L^2 norm terms of the initial state. This estimates ensures the unicity of the solution, and the existence may be proved again using the semi-discretized in space problem. However, in a fully 3D context, finding an energy estimate still remains an open question.

Conlusion

We are interested here in the study of PML equations for 3D acoustic wave propagation modeling in heterogeneous media. Basically, these equations are similar to the linearized Euler equations without advection term. These latter equations have been studied in a 2D context of homogeneous media, where wave velocity and acoustic impedance are constant functions.

Here, we obtain energy estimates of the solution for a 3D heterogeneous medium, where wave velocity impedance and acoustic impedance depend on space variable. These estimates are obtained in three different cases, arising in seismic wave modeling: the Cauchy problem, propagation in the half-space with a non homogeneous Dirichlet boundary condition, and transmission problem between two half-spaces separated by an acoustic impedance discontinuity.

In the two first cases, we find energy estimates that show that if the initial or the boundary condition is regular enough, then the H^1 norm of the solution remains bounded on time the intervalle $[0, T]$, with a potential exponential growth. In the third case, we come to the same conclusion assuming that the acoustic impedance depends only on the spatial direction normal to the interface. In each case, the energy estimates prove the unicity of the solution.

Finally, some problems remain to be investigated: first, in the latter case, the technique we used does not allow to conclude in the general 3D context. Furthermore, in the three cases, the existence question should be adressed. This may be done successfully by extending the method of Petit-Bergez on Maxwell equations in homogeneous media [10] based upon a semi-discretization in space of the problem.

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